

MINISTRY OF EDUCATION, SINGAPORE  
in collaboration with  
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION  
General Certificate of Education Advanced Level  
Higher 2

CANDIDATE  
NAME

Mr. Lim

CENTRE  
NUMBER

S

INDEX  
NUMBER

## MATHEMATICS

9758/01

Paper 1

October/November 2023

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

### READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.

Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

DO NOT WRITE ON ANY BARCODES.

Answer all the questions.

Write your answers in the spaces provided in the Question Paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

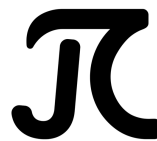
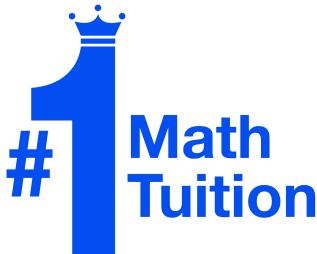
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Solution served as a suggestion only



where transFORMAtion begins®

BLANK PAGE



DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN



- 1 Find the exact equation of the tangent to the curve

$$\ln y = (11 - 5x)^2$$

at the point where  $x = 2$ .

[5]

$$\frac{1}{y} \left( \frac{dy}{dx} \right) = 2(11 - 5x)(-5)$$

$$\left. \frac{dy}{dx} \right|_{x=2} = 2(11 - 5(2))^3 (-5)(e)$$

$$= -10e$$

$$x=2, \quad y=e$$

$$y - e = -10e(x - 2)$$

$$y = -10ex + 21e \quad \Leftarrow$$

- 2 The first four terms of a sequence are given by  $u_1 = 10$ ,  $u_2 = 61$ ,  $u_3 = 206$  and  $u_4 = 469$ . It is given that  $u_n$  is a cubic polynomial.

(a) Find  $u_n$  in terms of  $n$ .

[3]

$$\text{let } u_n = ax^3 + bx^2 + cx + d$$

$$\begin{array}{l} n=1 \quad 10 = a + b + c + d \\ n=2 \quad 61 = 8a + 4b + 2c + d \\ n=3 \quad 206 = 27a + 9b + 3c + d \\ n=4 \quad 469 = 64a + 16b + 4c + d \end{array} \quad \left. \vphantom{\begin{array}{l} n=1 \\ n=2 \\ n=3 \\ n=4 \end{array}} \right\} \begin{array}{l} a = 4 \quad b = 23 \\ c = -46 \quad d = 29 \end{array}$$

$$u_n = 4x^3 + 23x^2 - 46x + 29$$

(b) Find the range of values of  $n$  for which  $u_n$  is greater than 25000.

[2]

$$4x^3 + 23x^2 - 46x + 29 > 25000$$

$$4x^3 + 23x^2 - 46x - 24971 > 0$$

$$n \geq 17 \quad \therefore n \in \mathbb{Z}^+$$



- 3 Vectors  $\mathbf{a}$  and  $\mathbf{b}$  are such that  $\mathbf{a} \cdot \mathbf{b} = -1$ . It is also given that  $(\mathbf{a} \times \mathbf{b} + \mathbf{a})$  is perpendicular to  $(\mathbf{a} \times \mathbf{b} + \mathbf{b})$ .

(a) Show that  $|\mathbf{a} \times \mathbf{b}| = 1$ .

[3]

$$(\mathbf{a} \times \mathbf{b} + \mathbf{a}) \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b}) = 0$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} = 0$$

$$|\mathbf{a} \times \mathbf{b}|^2 - 1 = 0$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = 1 \quad \square$$

(b) Hence find the angle between the direction of  $\mathbf{a}$  and the direction of  $\mathbf{b}$ .

[3]

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \quad \sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}$$

$$\cos \theta = \frac{-1}{\frac{1}{\sin \theta}}$$

$$\tan \theta = -1$$

$$\theta = \frac{3\pi}{4} \quad \square$$

- 4 (a) Find  $\int \cos px \cos qx \, dx$ , where  $p$  and  $q$  are constants such that  $p \neq q$  and  $p \neq -q$ . [2]

$$\begin{aligned} &= \int \frac{1}{2} (\cos(p+q)x + \cos(p-q)x) \, dx \\ &= \frac{\sin(p+q)x}{2(p+q)} + \frac{\sin(p-q)x}{2(p-q)} + c \end{aligned}$$

- (b) Given that  $n \neq 0$ , show that  $\int x \cos nx \, dx = \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + c$ , where  $c$  is an arbitrary constant. [3]

$$\begin{aligned} \int x \cos nx \, dx &= \frac{x}{n} \sin nx - \int \frac{1}{n} \sin nx \, dx \\ &= \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + c \end{aligned}$$

$$\begin{aligned} u &= x & \frac{dv}{dx} &= \cos nx \\ \frac{du}{dx} &= 1 & v &= \frac{1}{n} \sin nx \end{aligned}$$



- (c) Using the result in part (b) show that, for all positive integers  $n$ , the value of  $\int_0^{\pi} x \cos nx \, dx$  can be expressed as  $\frac{k}{n^2}$ , where the possible value(s) of  $k$  are to be determined. [2]

$$\int_0^{\pi} x \cos nx \, dx = \left[ \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} = \begin{cases} -\frac{1}{n^2} & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$k = 0 \text{ or } -2$$

- (d) Using the result in part (b) find the exact value of  $\int_0^{\pi/2} |x \cos 2x| \, dx$ . [3]

$$\begin{aligned} \int_0^{\pi/2} |x \cos 2x| \, dx &= \left[ \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right]_0^{\pi/4} + \left[ -\frac{x \sin 2x}{2} - \frac{\cos 2x}{4} \right]_{\pi/4}^{\pi/2} \\ &= \left[ \left( \frac{\pi}{8} + 0 \right) - \left( \frac{1}{4} \right) \right] + \left[ \frac{1}{4} - \left( -\frac{\pi}{8} + 0 \right) \right] \\ &= \frac{\pi}{4} \end{aligned}$$

- 5 (a) Use the method of differences to show that  $\sum_{r=2}^n \ln \left[ \frac{(r-1)(r+1)}{r^2} \right] = \ln \left( \frac{n+1}{n} \right) - \ln 2$ . [3]

$$\sum_{r=2}^n \ln(r-1) + \ln(r+1) - 2 \ln(r)$$

$$\begin{aligned}
 & \ln 1 + \ln 3 - 2 \ln 2 \\
 & + \ln 2 + \ln 4 - 2 \ln 3 \\
 & + \ln 3 + \ln 5 - 2 \ln 4 \\
 & + \ln 4 + \ln 6 - 2 \ln 5 \\
 & \vdots \\
 & + \ln(n-3) + \ln(n-1) - 2 \ln(n-2) \\
 & + \ln(n-2) + \ln n - 2 \ln(n-1) \\
 & + \ln(n-1) + \ln(n+1) - 2 \ln n
 \end{aligned}$$

$$\begin{aligned}
 & = -\ln 2 + \ln(n+1) - \ln n \\
 & = \ln \left( \frac{n+1}{n} \right) - \ln 2
 \end{aligned}$$

(b) Show that the corresponding infinite series is convergent and state the sum to infinity. [2]

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{r=2}^n \ln \left( \frac{(r-1)(r+1)}{r^2} \right) &= \lim_{n \rightarrow \infty} \ln \left( 1 + \frac{1}{n} \right) - \ln 2 \\ &= \ln 1 - \ln 2 \\ &= -\ln 2 \end{aligned}$$

Sum to infinity :  $-\ln 2$

(c) Show that  $\sum_{r=10}^{20} \ln \left[ \frac{(r-1)(r+1)}{r^2} \right] = \ln \left( \frac{a}{b} \right)$ , where  $a$  and  $b$  are integers to be found. [2]

$$\begin{aligned} \sum_{r=10}^{20} \ln \left[ \frac{(r-1)(r+1)}{r^2} \right] &= \sum_{r=1}^{20} \ln \left[ \frac{(r-1)(r+1)}{r^2} \right] - \sum_{r=1}^9 \ln \left[ \frac{(r-1)(r+1)}{r^2} \right] \\ &= \ln \frac{21}{20} - \ln 2 - \left( \ln \frac{10}{9} - \ln 2 \right) \\ &= \ln \frac{189}{200} \end{aligned}$$

$$\therefore a = 189 \quad b = 200 \quad \#$$

- 6 (a) Using double angle formulae, show that  $\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3)$ .

$$\begin{aligned}\cos^4 \theta &= \left(\frac{\cos 2\theta + 1}{2}\right)^2 \\ &= \frac{\cos^2 2\theta + 2\cos 2\theta + 1}{4} \\ &= \frac{1}{4} \left[ \frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1 \right] \\ &= \frac{1}{8} [\cos 4\theta + 4\cos 2\theta + 3] \quad \# \end{aligned}$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\frac{\cos 2A + 1}{2} = \cos^2 A$$

- (b) The region  $R$  lies in the first quadrant and is bounded by the curve  $y^4 = (9 - x^2)^3$ , the  $x$ -axis and the lines  $x = 1.5$  and  $x = 3$ .  $R$  is rotated about the  $x$ -axis through  $2\pi$  radians. Using the substitution  $x = 3 \sin \theta$ , find the exact volume generated. [6]

$$\begin{aligned}\text{Vol} &= \int_{1.5}^3 \pi y^2 dx \\ &= \int_{\pi/6}^{\pi/2} \pi (9 - 9\sin^2 \theta)^{3/2} 3 \cos \theta d\theta \\ &= 3\pi \int_{\pi/6}^{\pi/2} 9^{3/2} \cos^4 \theta d\theta \\ &= \frac{81\pi}{8} \int_{\pi/6}^{\pi/2} (\cos 4\theta + 4\cos 2\theta + 3) d\theta \\ &= \frac{81\pi}{8} \left[ \frac{\sin 4\theta}{4} + \frac{4\sin 2\theta}{2} + 3\theta \right]_{\pi/6}^{\pi/2} \\ &= \frac{81\pi}{8} \left( \frac{3\pi}{2} - \left( \frac{\sqrt{3}}{8} + \sqrt{3} + \frac{\pi}{2} \right) \right) \\ &= \frac{81\pi}{8} \left( \pi - \frac{9}{8}\sqrt{3} \right) \quad \# \end{aligned}$$

$$x = \frac{3}{2} \quad \theta = \pi/6$$

$$x = 3 \quad \theta = \pi/2$$

$$x = 3 \sin \theta$$

$$\frac{dx}{d\theta} = 3 \cos \theta$$

DO NOT WRITE IN THIS MARGIN  
DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN  
DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN  
DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN  
DO NOT WRITE IN THIS MARGIN

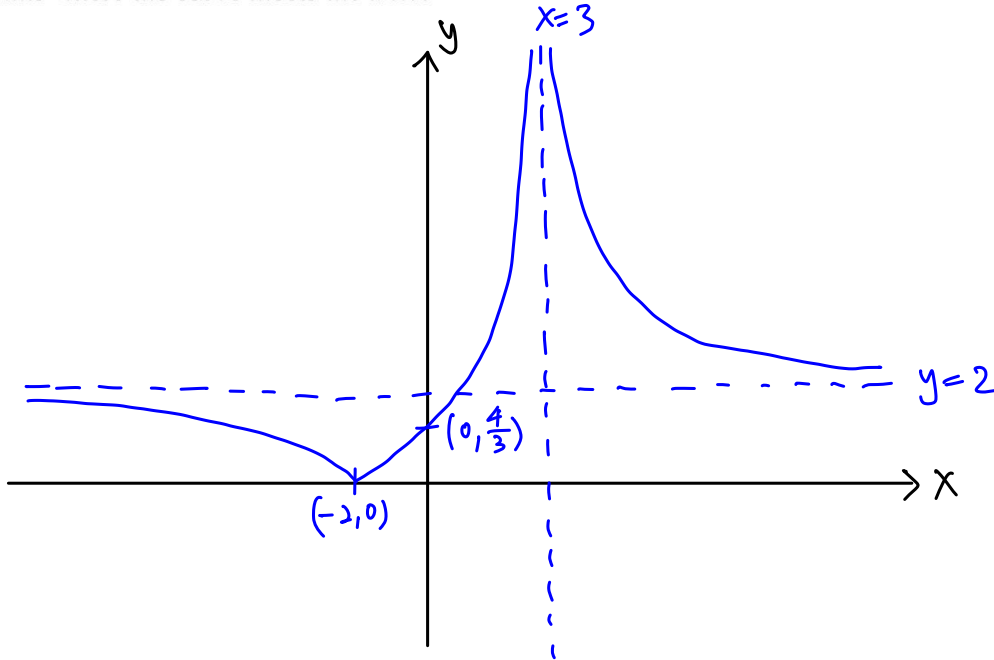
DO NOT WRITE IN THIS MARGIN  
DO NOT WRITE IN THIS MARGIN

6 [Continued]

7 The function  $f$  is defined by

$$f: x \mapsto \left| \frac{2x+4}{3-x} \right|, \quad x \in \mathbb{R}, \quad x \neq 3.$$

- (a) Sketch the graph of  $y = f(x)$ , giving the equations of any asymptotes and the coordinates of the points where the curve meets the axes. [3]



- (b) Hence state the range of  $f$ .

$$R_f: [0, \infty)$$

- (c) Explain why the function  $f^2$  does not exist.

$$D_f: (-\infty, 3) \cup (3, \infty)$$

$$R_f \neq D_f$$



The domain of  $f$  is further restricted to  $x \leq a$ , where  $a$  is a constant.

(d) State the greatest value of  $a$  such that the function  $f^{-1}$  exists.

[1]

$$a = -2$$

(e) Hence find  $f^{-1}(x)$  and state its domain.

[4]

$$y = -\frac{2x+4}{3-x}$$

$$-y(3-x) = 2x+4$$

$$xy - 3y = 2x+4$$

$$x(y-2) = 4+3y$$

$$x = \frac{4+3y}{y-2}$$

$$f^{-1}(x) = \frac{4+3x}{x-2}$$

$$D_{f^{-1}} = [0, 2)$$





- (b) Find the complex numbers  $v$  and  $w$  which satisfy the following simultaneous equations.

$$2v + |w| = 1 \quad (1)$$

$$3v - iw = -3 + 4i \quad (2)$$

Give your answers in the form  $a + ib$ , where  $a$  and  $b$  are real numbers.

[5]

$$\text{Let } v = a, \quad a \in \mathbb{R}$$

$$(2) \quad -iw = -3 - 3a + 4i$$

$$iw = 3 + 3a - 4i$$

$$|iw| = \sqrt{9 + 18a + 9a^2 + 16}$$

$$\therefore |w| = \sqrt{9a^2 + 18a + 25}$$

$$(1) \quad |w| = 1 - 2a$$

$$\Rightarrow \sqrt{9a^2 + 18a + 25} = 1 - 2a$$

$$9a^2 + 18a + 25 = 1 - 4a + 4a^2$$

$$5a^2 + 22a + 24 = 0$$

$$(a+2)(5a+12) = 0$$

$$a = -2 \quad \text{or} \quad -\frac{12}{5}$$

$$v = -2$$

$$v = -\frac{12}{5}$$

$$w = -4 + 3i$$

$$w = -4 + \frac{21}{5}i$$

- 9 The line  $l_1$  contains the point  $A$  with coordinates  $(-3, 1, 2)$  and is parallel to the vector  $\begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$ , where  $a$  is a constant. The line  $l_2$  has equation  $\frac{x+2}{3} = \frac{y-1}{2} = z-5$ . It is given that  $l_1$  and  $l_2$  cross at the point  $B$ .

(a) Find the value of  $a$  and the coordinates of  $B$ . [5]

$$l_1: r = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix} \quad \text{where } \alpha \in \mathbb{R}$$

$$l_2: r = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{where } \lambda \in \mathbb{R}$$

$$-3 + 2\alpha = -2 + 3\lambda \Rightarrow 2\alpha - 3\lambda = 1$$

$$1 + \alpha = 1 + 2\lambda \Rightarrow \alpha = 2\lambda$$

$$\therefore \lambda = 1 \quad \alpha = 2$$

$$2 + 2a = 6$$

$$a = 2$$

$$B(1, 3, 6) \quad \#$$

(b) The plane  $\pi_1$  contains the point  $A$  and is perpendicular to  $l_2$ .

(i) Find the shortest distance from  $B$  to  $\pi_1$ . [3]

$$\pi_1: r \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = -5$$

let  $\pi_2$  be plane containing point  $B$  and parallel to  $\pi_1$ .

$$\pi_2: r \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 15$$

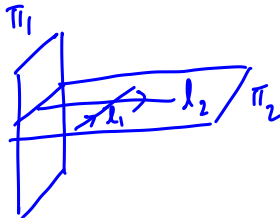
$$\therefore \text{Dist} = \frac{20}{\sqrt{14}} = \frac{10\sqrt{14}}{7} \quad \#$$

(ii) Hence find the acute angle between  $l_1$  and  $\pi_1$ . [2]

$$\sin \theta = \frac{\begin{vmatrix} 2 \\ 1 \\ 2 \end{vmatrix} \cdot \begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix}}{\begin{vmatrix} 2 \\ 1 \\ 2 \end{vmatrix} \begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix}}$$

$$\theta = 62.98^\circ \approx 63.0^\circ$$

(c) The plane  $\pi_2$  is perpendicular to  $\pi_1$  and contains  $l_1$ . Find a cartesian equation of  $\pi_2$ . [3]



$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-4 \\ -(2-6) \\ 4-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$$

$$\therefore \pi_2: r \cdot \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = 15$$

$$\pi_2: -3x + 4y + z = 15 \quad \#$$



As part of a new health plan, Andrew fixes his energy intake at 80% of the value found in part (b).

- (c) By solving the differential equation in part (a), show that Andrew's mass while he is on the plan satisfies the equation  $M = 88 + 22e^{-30kt}$ . [4]

$$C = 2640$$

$$\begin{aligned} \frac{dm}{dt} &= k(2640 - 30M) \\ &= 30k(88 - M) \end{aligned}$$

$$\int \frac{1}{88 - M} dM = \int 30k dt$$

$$-\ln|88 - M| = 30kt + C$$

$$88 - M = Ae^{-30kt} \quad \text{where } A = \pm e^C$$

$$t = 0 \quad M = 110 \quad A = -22$$

$$88 - M = -22e^{-30kt}$$

$$M = 88 + 22e^{-30kt} \quad * \text{ Shown}$$



- 11 Wei is saving for a property that she intends to buy in the future. Wei needs to save a minimum of \$50 000. She saves regularly in an account which offers no interest. She makes an initial deposit on 31 January 2021 of \$ $a$ . Each subsequent month, she deposits \$50 more than she deposited in the previous month. Her final deposit is made on 31 December 2023.

(a) Find, to the nearest cent, the smallest value of  $a$  so that she saves at least \$50 000. [2]

$$S_{36} = \frac{36}{2}(2a + 35(50)) > 50000$$

$$a > \$513.88$$

Wei has arranged to purchase a property on 1 January 2024 with the aid of a loan of \$400 000. The terms of the loan are that interest of 0.1% is added to the amount owing at the start of every month, with the first interest amount added on 1 January 2024. Wei makes a monthly repayment of \$ $x$  at the end of every month, with the first repayment on 31 January 2024.

(b) Show that the amount, in dollars, that Wei owes at the end of  $n$  months is

$$400000 \times 1.001^n - 1000x(1.001^n - 1). \quad [3]$$

$n$	Start	End
1	400000(1.001)	400000(1.001) - $x$
2	$[400000(1.001) - x](1.001)$	$[400000(1.001) - x](1.001) - x$ $= 400000(1.001)^2 - x(1 + 1.001)$
$\vdots$	$\vdots$	$\vdots$
$n$		$400000(1.001)^n - x(1 + 1.001 + \dots + 1.001^{n-1})$

$$\begin{aligned} \text{Amt owed: } & 400000(1.001)^n - x \left[ \frac{1.001^n - 1}{1.001 - 1} \right] \\ & = 400000(1.001)^n - 1000x(1.001^n - 1) \quad \parallel \end{aligned}$$

## 11 [Continued]

- (c) (i) Suppose that the loan is repaid in 360 monthly repayments. Find the value of the monthly repayment to the nearest cent. [2]

$$400000(1.001)^{360} - 1000x(1.001^{360} - 1) = 0$$

$$x = \$1323.63$$

- (ii) Use your answer to part (c)(i) to calculate the total interest paid on the loan in this case. [1]

$$\text{Int: } \$76508.52$$

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

- (d) (i) Now suppose Wei decides to repay the loan at the rate of \$1600 per month for  $k$  months, plus a final monthly repayment of  $y$ , where  $y < 1600$ . Find  $k$  and  $y$ . [4]

$$400000(1.001)^k - 1000(1600)(1.001^k - 1) = 0$$

$$\text{USE G.C. , } k = 287$$

$$y = \$1320.22 \times 1.001$$

$$= \$1321.53$$

- (ii) Hence find the total saving to Wei by making the repayments in part (d)(i) rather than the repayments in part (c)(i), giving your answer to 4 significant figures. [1]

$$\$15988.3 \approx \$15990 \quad (4sf)$$

**BLANK PAGE**



DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

DO NOT WRITE IN THIS MARGIN

Permission to reproduce items where third-party owned material protected by copyright is included has been sought and cleared where possible. Every reasonable effort has been made by the publisher (UCLES) to trace copyright holders, but if any items requiring clearance have unwittingly been included, the publisher will be pleased to make amends at the earliest possible opportunity.

Cambridge Assessment International Education is part of Cambridge Assessment. Cambridge Assessment is the brand name of the University of Cambridge Local Examinations Syndicate (UCLES), which is a department of the University of Cambridge.

