1 a) $u_2 = 3a+5$ $\quad u_3 = 3(3a+5)+5$ $\quad u_4 = 27a+65$

$= 9a+20$

b) $\sum_{r=1}^{4} u_r = 110$

$a + 3a+5 + 9a+20 + 27a+65 = 110$

$40a = 20$

$a = 1/2$

2 a) $\frac{20}{2}(2a+19d) = 65 \Rightarrow 2a+19d = 6.5$

$\frac{28}{2}(2a+27d) - 65 = -30 \Rightarrow 2a+27d = 2.5$

$\left.\right\}$ $d = -\frac{1}{2}$ $\quad a = 8$

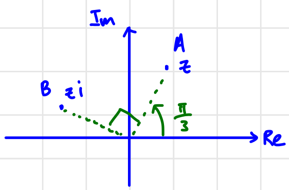
b) $u_{n+1} = e^{k(n+2)}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{e^{k(n+2)}}{e^{k(n+1)}} = e^{kn+2k-kn-k} = e^k$ constant $\therefore$ G.P.

3 a) $|z| = \left|\frac{\sqrt{3}+i}{\sqrt{3}-i}\right| = 1$ $\quad \arg z = \arg(\sqrt{3}+i) - \arg(\sqrt{3}-i)$

$= \frac{\pi}{3}$

b)



solutions served as a suggestion only

4 a) $\frac{dy}{dx} = \frac{3\sqrt{x+1} - \frac{1}{2}(x+1)^{-1/2}(3x+10)}{x+1} = \frac{3}{\sqrt{x+1}} - \frac{3x+10}{2(x+1)^{3/2}}$

$\frac{dy}{dx} = 0 \Rightarrow 3\sqrt{x+1} = \frac{3x+10}{2\sqrt{x+1}}$

$6x+6 = 3x+10 \quad \therefore x = \frac{4}{3}$

b) $\frac{d^2y}{dx^2} = -\frac{3}{2(x+1)^{3/2}} - \left(\frac{6(x+1)^{3/2} - 3(x+1)^{1/2}(3x+10)}{4(x+1)^3} \right)$

$\left.\frac{d^2y}{dx^2}\right|_{x=4/3} = 0.4208 \approx 0.421 > 0 \quad \therefore$ minimum point

5 a) $e^x - 2e^{-x} - 1 \geqslant 0$

$e^{2x} - e^x - 2 \geqslant 0$

$(e^x - 2)(e^x + 1) \geqslant 0$

$e^x \leq -1$ or $e^x \geqslant 2$

rej $\quad x \geqslant \ln 2$ *

b) $\int_0^1 |e^x - 2e^{-x} - 1| \, dx$

$= \int_0^{\ln 2} -e^x + 2e^{-x} + 1 \, dx + \int_{\ln 2}^1 e^x - 2e^{-x} - 1 \, dx$

$= \left[-e^x - 2e^{-x} + x \right]_0^{\ln 2} + \left[e^x + 2e^{-x} - x \right]_{\ln 2}^1$

$= \left(-2 - 2\left(\frac{1}{2}\right) + \ln 2\right) - (-1-2) + \left(e + 2e^{-1} - 1\right) - \left(2 + 2\left(\frac{1}{2}\right) - \ln 2\right)$

$= -4 + e + 2e^{-1} + 2\ln 2$ *

6 a) $\dfrac{dM}{dt} = -kM$

$\int \frac{1}{m}\, dM = \int -k\, dt$

$\ln|m| = -kt + c$

$M = Ae^{-kt}$ where $A = e^{\pm c}$

when $t = 0$ $M = M_0$ So $A = M_0$ $\Rightarrow M = M_0 e^{-kt}$

when $t = 5730$ $M = \frac{1}{2} M_0$ $\Rightarrow$ $\frac{1}{2} = e^{-5730k}$

$\therefore k = 0.000120968$
≈ 0.000121

b) $0.2 = e^{-0.000120968t}$
$t = 13304.658$
≈ 13300 years (nearest 100)

7 a) $\dfrac{d}{dx}(x\ln x - x) = \ln x + 1 - 1 = \ln x$

b) Vol: $\pi \int_1^e (\ln x)^2\, dx$

$u = (\ln x)^2$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2\ln x}{x}$ $\quad v = x$

$= \pi \left\{ \left[x(\ln x)^2 \right]_1^e - 2\int_1^e \ln x\, dx \right\}$

$= \pi \left\{ e - 2\left[x\ln x - x \right]_1^e \right\}$

$= \pi (e - 2)$ units^3 #

7 c) $\text{Vol} = \pi e^2 - \pi \int_0^1 e^{2y}\, dy = 13.177 \approx 13.18 \text{ units}^3$

8 a) Let line ℓ be $r = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \ \lambda \in \mathbb{R}$

$$\begin{pmatrix} -3+2\lambda \\ 1+\lambda \\ 3\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 2 \quad \Rightarrow -6+4\lambda+1+\lambda+9\lambda = 2$$

$$14\lambda = 7$$

$$\lambda = \tfrac{1}{2}$$

$$\therefore \ P: \begin{pmatrix} -3+1 \\ 1+\frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$$

b) $\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + \overrightarrow{OP_R}$

$$\therefore \ \overrightarrow{OP_R} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

c) $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3-10 \\ -(6-10) \\ 4+2 \end{pmatrix} = \begin{pmatrix} -13 \\ 4 \\ 6 \end{pmatrix}$

$$\cos\theta = \frac{\left| \begin{pmatrix} -13 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right|}{\left| \begin{matrix} -13 \\ 4 \\ 6 \end{matrix} \right| \left| \begin{matrix} 2 \\ 1 \\ 3 \end{matrix} \right|} = \frac{|-4|}{\sqrt{221}\sqrt{14}}$$

$$\theta = 85.9^\circ$$

9 a) $\sin y = px$

$\cos y \frac{dy}{dx} = p$

$\Rightarrow \frac{dy}{dx} = p\sec y$ *

b) $\frac{d^2y}{dx^2} = p\sec y\tan y \frac{dy}{dx} = p^2\sec^2 y \tan y$

$\cos^2 y \frac{d^2y}{dx^2} = p^2 \tan y$

$\Rightarrow -2\cos y \sin y \frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) + \cos^2 y\left(\frac{d^3y}{dx^3}\right) = p^2 \sec^2 y \frac{dy}{dx}$

when $x=0$ $\quad y=0$

$\frac{dy}{dx} = p \qquad \frac{d^2y}{dx^2} = 0 \qquad \frac{d^3y}{dx^3} = p^3$

$\therefore y = px + \frac{p^3}{6}x^3 + \ldots$

c) $y = \sin^{-1} 3x$

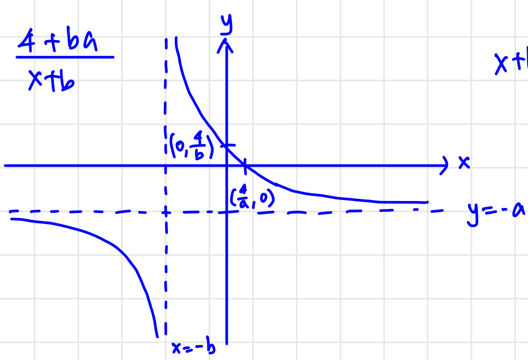
$\frac{dy}{dx} = \frac{3}{\sqrt{1-(3x)^2}} = \frac{3}{\sqrt{1-9x^2}}$

from (b) $y = 3x + \frac{1}{2}x^2 + \frac{9}{2}x^3 + \ldots$

$\frac{dy}{dx} = 3 + x + \frac{27}{2}x^2 + \ldots$

$\Rightarrow \frac{1}{\sqrt{1-9x^2}} = 1 + \frac{9}{2}x^2 + \ldots$

10 a) $y = -a + \frac{4+ba}{x+b}$



$$\begin{array}{r} -a \\ x+b \overline{)\,-ax+4} \\ -(-ax-ba) \\ \hline 4+ba \end{array}$$

b) $y = -a + \frac{4+ba}{x+b}$

①: sub $y = y-a$ $\Rightarrow$ $y - a = -a + \frac{4+ba}{x+b}$

$y = \frac{4+ba}{x+b}$

Translation a units in positive y-axis direction

②: sub $y = y(4+ba)$ $\Rightarrow$ $y(4+ba) = \frac{4+ba}{x+b}$

$y = \frac{1}{x+b}$

Scaling parallel to y-axis by factor $\frac{1}{4+ba}$

③: sub $x = x-b$ $\Rightarrow$ $y = \frac{1}{x}$

Translation b units in positive x-axis direction

c) $y(x+b) = 4 - ax$

$yx + ax = 4 - yb$

$x = \frac{4-yb}{y+a}$

$\therefore f^{-1} : x \mapsto \frac{4-xb}{x+a}$, $x \in \mathbb{R}$ $x \neq -a$

$R_{f^{-1}} = D_f$: $x \in \mathbb{R} \setminus \{-b\}$

d) $f^{-1} = f$

$\Rightarrow f^{2025}(2) = f^{2024} f(2)$

$= f(2)$

$= \frac{4-2a}{2+a}$ #

solutions served as a suggestion only

11 a) $x = 28(\theta\cos\theta + 1)$

$\frac{dx}{d\theta} = 28\left(\cos\theta - \theta\sin\theta\right)$

$y = 20(2 - \theta\sin\theta)$

$\frac{dy}{d\theta} = 20\left(-(\sin\theta + \theta\cos\theta)\right)$

$= -20(\sin\theta + \theta\cos\theta)$

$$\frac{dy}{dx} = \frac{-5(\sin\theta + \theta\cos\theta)}{7(\cos\theta - \theta\sin\theta)}$$

b) (i) $\cos\theta = \theta\sin\theta$

$\theta = -0.86033$ or 0.86033

≈ -0.86 ≈ 0.86

(ii) $x = 12.289$ $x = 43.710$

width : 31.4216

≈ 31.4 m

c (i) $\theta = 0$

(ii) $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$

(iii) $40 - 8.584 = 31.4159 \approx 31.4$

d) $AE = 40 - 28.778 = 11.2219$

$\frac{1}{2}CD = 15.7108$

Safety range$(14.1, 17.28)$

Ans: No.

12 a) $\because$ coefficients are all real, complex roots occurs in conjugate pairs.

other root : $1-2i$

b) $z^3+pz^2+qz-10 = (z-(1+2i))(z-(1-2i))(z-a)$

$= [(z-1)^2-(2i)^2][z-a]$

$= (z^2-2z+5)(z-a)$

$\therefore a=2$ $\quad\therefore$ 3rd root is 2.

$(z^2-2z+5)(z-2) = z^3-2z^2-2z^2+4z+5z-10$

$= z^3-4z^2+9z-10$

$\therefore p=-4 \quad q=9$

c) $z^3+pz^2+qz-10=0$

$1+\frac{p}{z}+\frac{q}{z^2}-\frac{10}{z^3}=0$

let $w=\frac{1}{z}$

$1+pw+qw^2-10w^3=0$

$\therefore w=\frac{1}{1+2i}=\frac{1}{5}-\frac{2}{5}i \qquad w=\frac{1}{1-2i}=\frac{1}{5}+\frac{2}{5}i \qquad w=\frac{1}{2}$ //

1 A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = a,$$

$$u_{r+1} = 3u_r + 5, \quad r \geqslant 1,$$

where a is a constant.

(a) Find expressions for u_2 and u_3 in terms of a. [2]

(b) It is given that $\sum_{r=1}^{4} u_r = 110$. Find the value of a. [3]

2 (a) The sum of the first 20 terms of an arithmetic sequence is 65. The sum of the next 8 terms is -30.

Find the first term and the common difference of this sequence. [3]

(b) A sequence has nth term given by $u_n = \mathrm{e}^{k(n+1)}$, where k is a constant and $k \neq 0$.

Show that this sequence is geometric. [2]

3 **Do not use a calculator in answering this question.**

A complex number z is given by $z = \dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$.

(a) Find $|z|$ and $\arg(z)$. [4]

The point A represents the complex number z on an Argand diagram. The point A is rotated anticlockwise through $\frac{1}{2}\pi$ about the origin O to the point B.

(b) Plot the points A and B on an Argand diagram. State in terms of z the complex number that is represented by B, explaining your answer. [3]

4 A curve is given by the equation $y = \dfrac{3x+10}{\sqrt{x+1}}$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and hence find the x-coordinate of the turning point of the curve. [3]

(b) Use calculus to find the value of the second derivative at the turning point and state the nature of this turning point. [3]

5 (a) Solve exactly the inequality $\mathrm{e}^x \geqslant 2\mathrm{e}^{-x} + 1$. [3]

(b) Hence find $\int_0^1 |\mathrm{e}^x - 2\mathrm{e}^{-x} - 1|\,\mathrm{d}x$, giving your answer in exact form. [4]

6 The carbon content of plants includes a small proportion of the radioactive isotope carbon-14. The mass of carbon-14 in dead plants decreases over time. Scientists measure the mass of carbon-14 in plants to estimate how long ago they died.

The mass of carbon-14 in dead plants decreases by 50% in 5730 years.

The mass, M, of carbon-14 in a plant t years after it has died is modelled by the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -kM,$$

where k is a positive constant. The initial value of M is M_0.

(a) By solving the differential equation, find the value of k. [4]

(b) A dead plant has 20% of its original mass of carbon-14 remaining. Determine, to the nearest 100 years, how long ago the plant died. [2]

7 **(a)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x - x) = \ln x$. [2]

A curve C is given by the equation $y = \ln x$. The region R is bounded by C, the x-axis and the line $x = \mathrm{e}$.

(b) Using the result in part **(a)**, find the exact volume generated when R is rotated 360° about the x-axis. [4]

(c) Find the volume generated when R is rotated 360° about the y-axis, giving your answer correct to 2 decimal places. [3]

8 The plane π_1 has equation $2x + y + 3z = 2$. A point P has position vector $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$.

(a) Find the foot of the perpendicular from P to the plane π_1. [4]

(b) Find the position vector of the reflection of P in the plane π_1. [2]

A second plane π_2 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$, where λ and μ are parameters.

(c) Find the acute angle between π_1 and π_2. [3]

9 It is given that $y = \sin^{-1} px$, where p is a constant.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = p \sec y$. [2]

(b) By further differentiation of the result in part **(a)** find, in terms of p, the Maclaurin series for y up to and including the term in x^3. [5]

(c) Use your answer to part **(b)** to find the series expansion of $\frac{1}{\sqrt{1-9x^2}}$, up to and including the term in x^2. [3]

10 The function f is given by

$$\mathrm{f}: x \mapsto \frac{4-ax}{x+b}, \qquad x \in \mathbb{R}, \qquad x \neq -b,$$

where a and b are positive constants.

(a) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the axes. [4]

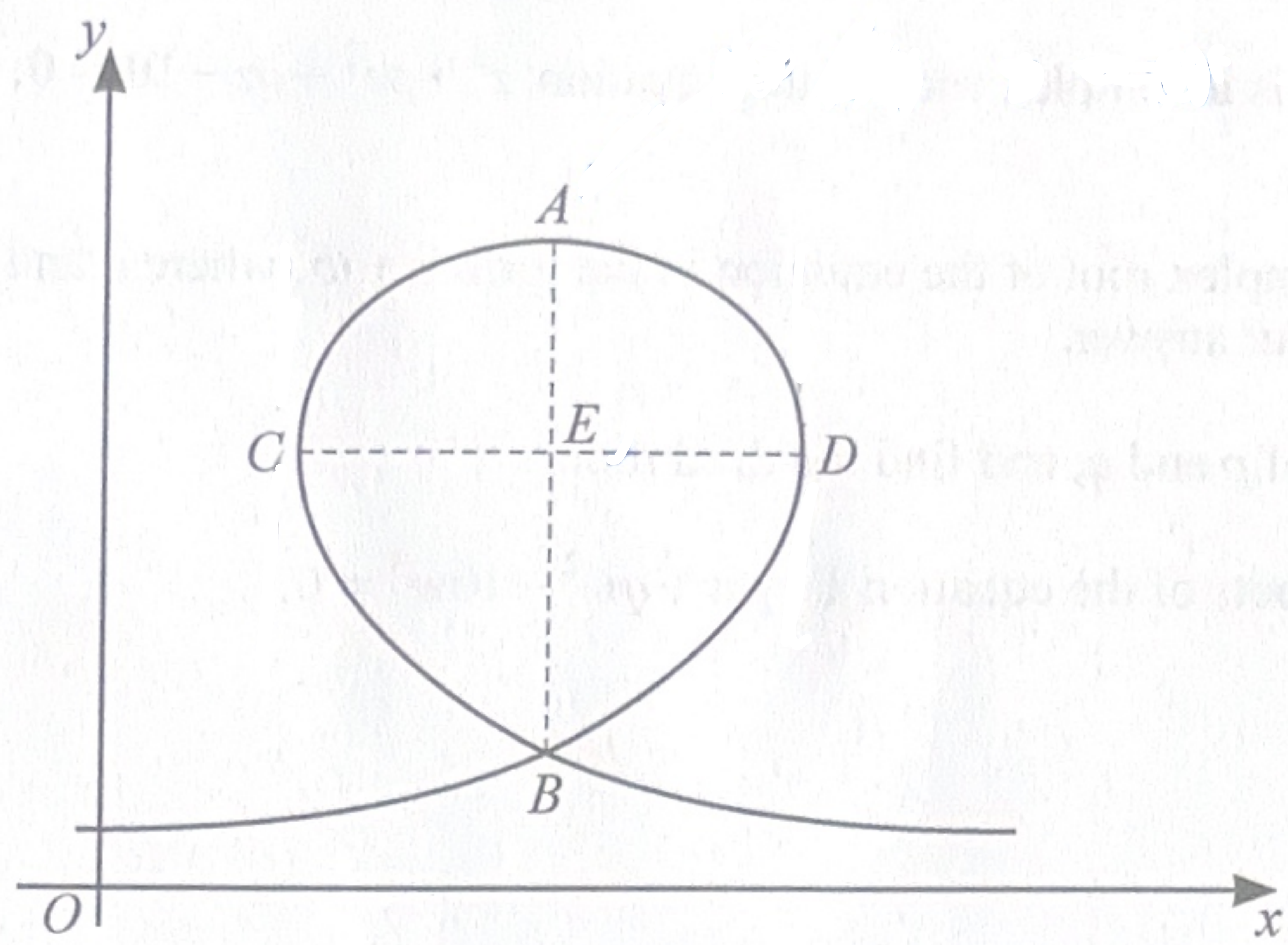
(b) Describe a sequence of 3 transformations that would transform the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \frac{1}{x}$. [4]

(c) Find $\mathrm{f}^{-1}(x)$ and state its range. [3]

It is now given that $b = a$.

(d) Find the value of the composite function $\mathrm{f}^{2025}(x)$ when $x = 2$, giving your answer in terms of a. [1]

11



Engineers are designing a roller coaster. The diagram shows the side view of a section of track with a loop. This section is modelled by the curve with parametric equations

$$x = 28(\theta\cos\theta + 1), \qquad y = 20(2 - \theta\sin\theta),$$

for $-\frac{2}{3}\pi \leqslant \theta \leqslant \frac{2}{3}\pi$, where x and y are measured in metres. The x-axis models the horizontal ground.

The maximum point on the curve is at A, and the line AB is vertical. Points C and D are where the tangent to the curve is parallel to the y-axis. Point E is the intersection of the lines AB and CD.

(a) Find $\frac{dy}{dx}$ in terms of θ. [3]

(b) **(i)** Find the values of θ at points C and D, giving your answers correct to 2 significant figures. [3]

(ii) Hence find the greatest width, CD, of the loop. [2]

The x-coordinate of point A is 28.

(c) **(i)** Find the value of θ at point A. [1]

(ii) Find the 2 possible values of θ at point B. [2]

(iii) Hence find the distance AB. [1]

Engineers know that, for safety reasons, the distance AE should be within 10% of the distance $\frac{1}{2}CD$.

(d) Determine whether the design of this section of track satisfies this condition. [2]

Question 12 is printed on the next page.

12 Do not use a calculator in answering this question.

It is given that $1+2\mathrm{i}$ is a complex root of the equation $z^3+pz^2+qz-10=0$, where p and q are real numbers.

(a) State another complex root of the equation in the form $a+\mathrm{i}b$, where a and b are real numbers and $b \neq 0$. Justify your answer. [2]

(b) Find the values of p and q, and find the third root. [6]

(c) Hence find the roots of the equation $1+pw+qw^2-10w^3=0$. [2]