Suggested Answers

MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME: Mr. Lim

CENTRE NUMBER: S

INDEX NUMBER:

MATHEMATICS **9758/02**

Paper 2 **October/November 2021**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Suggested Answers

Section A: Pure Mathematics [40 marks]

1 One of the roots of the equation $x^3 + 2x^2 + ax + b = 0$, where a and b are real, is $1 + \frac{1}{2}\mathrm{i}$. Find the other roots of the equation and the values of a and b. [5]

$\because$ all coefficients are real, by conjugate root thm, $1-\frac{1}{2}i$ is a root.

$$\left(x-(1-\tfrac{1}{2}i)\right)\left(x-(1+\tfrac{1}{2}i)\right)$$
$$=\left((x-1)+\tfrac{1}{2}i\right)\left((x-1)-\tfrac{1}{2}i\right)$$
$$=(x-1)^2-(\tfrac{1}{2}i)^2$$
$$=x^2-2x+1+\tfrac{1}{4}$$
$$=x^2-2x+\tfrac{5}{4}$$

so $(x^3+2x^2+ax+b)=(x-A)(x^2-2x+\frac{5}{4})$

x^2: $2=-2-A$

$A=-4$

$\therefore$ The roots are -4, $1-\frac{1}{2}i$, $1+\frac{1}{2}i$

$$(x+4)(x^2-2x+\tfrac{5}{4})$$
$$=x^3-2x^2+\tfrac{5}{4}x+4x^2-8x+5$$
$$=x^3+2x^2-\tfrac{27}{4}x+5$$

$a=-\frac{27}{4}$, $b=5$ #

1 [Continued]

where transFormAtion begins®

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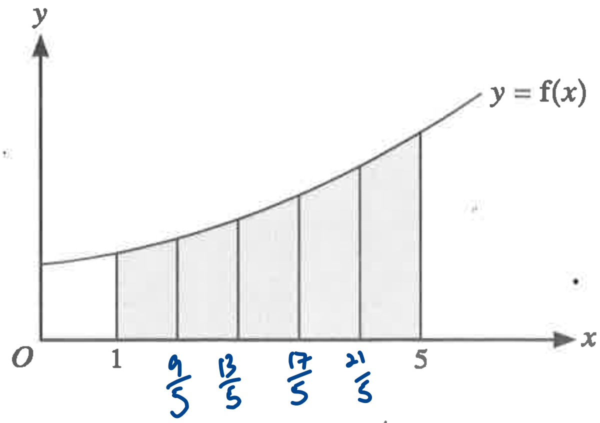
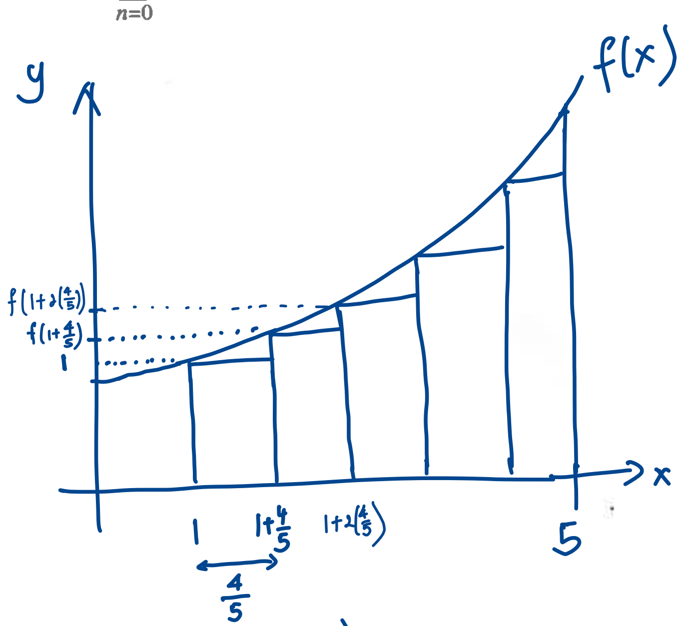
2 The diagram shows a sketch of the curve $y = f(x)$. The region under the curve between $x = 1$ and $x = 5$, shown shaded in the diagram, is A. This region is split into 5 vertical strips of equal width, h.



(a) State the value of h and show, using a sketch, that $\sum_{n=0}^{4}(f(1+nh))h$ is less than the area of A. [2]

$h = \frac{4}{5}$



Area of rectangles

$= \frac{4}{5}(f(1)) + \frac{4}{5}\left(f\left(1+\frac{4}{5}\right)\right) + \frac{4}{5}\left(f\left(1+2\left(\frac{4}{5}\right)\right)\right) + \ldots + \frac{4}{5}\left(f\left(1+4\left(\frac{4}{5}\right)\right)\right)$

$= \frac{4}{5}\left(f(1) + f\left(1+\frac{4}{5}\right) + f\left(1+2\left(\frac{4}{5}\right)\right) + \ldots + f\left(1+4\left(\frac{4}{5}\right)\right)\right)$

$= \frac{4}{5}\left(\sum_{n=0}^{4} f\left(1+\frac{4}{5}(n)\right)\right)$

$= \sum_{n=0}^{4} f(1+hn)\,h \quad \because h = \frac{4}{5}$

Area of rectangles is an underestimate of the area of A

Hence $\sum_{n=0}^{4} f(1+nh)h < A$.

(b) Find a similar expression that is greater than the area of A. [1]

$$\sum_{n=1}^{5} f(1+nh)h = \sum_{n=1}^{5} f\left(1+\tfrac{4}{5}n\right)\tfrac{4}{5}$$

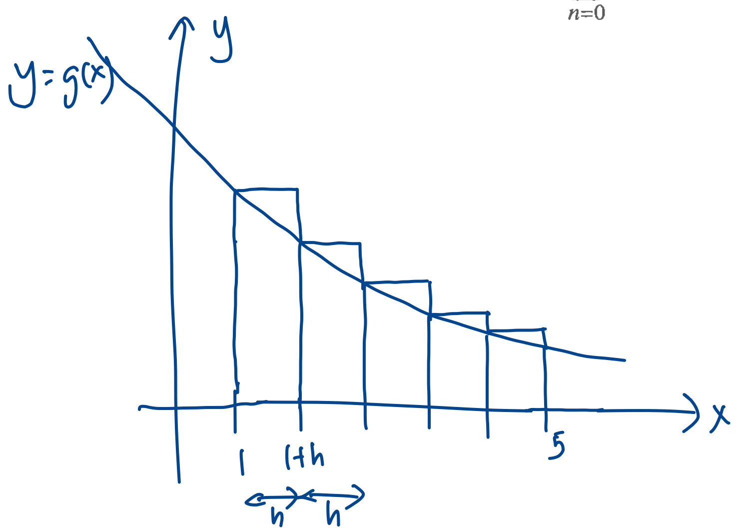
You are now given that $f(x) = \frac{1}{20}x^2 + 1$.

(c) Use the expression given in part **(a)** and your expression from part **(b)** to find lower and upper bounds for the area of A. [2]

$$\frac{4}{5}\sum_{n=0}^{4} f\left(1+\tfrac{4}{5}n\right)$$
$$= \frac{4}{5}\sum_{n=0}^{4} \frac{1}{20}\left(1+\tfrac{4}{5}n\right)^2 + 1$$
$$= \frac{1}{25}\sum_{n=0}^{4}\left(1+\tfrac{4}{5}n\right)^2 + 5\left(\tfrac{4}{5}\right)$$
$$= \frac{1}{25}\left(1^2+\left(\tfrac{9}{5}\right)^2+\left(\tfrac{13}{5}\right)^2+\left(\tfrac{17}{5}\right)^2+\left(\tfrac{21}{5}\right)^2\right)+4$$
$$= \frac{1}{25}\left(\frac{201}{5}\right)+4$$
$$= \frac{701}{125}$$

$$\frac{4}{5}\sum_{n=1}^{5} f\left(1+\tfrac{4}{5}n\right)$$
$$= \frac{4}{5}\sum_{n=1}^{5} \frac{1}{20}\left(1+\tfrac{4}{5}n\right)^2 + 1$$
$$= \frac{1}{25}\left(\left(\tfrac{9}{5}\right)^2+\left(\tfrac{13}{5}\right)^2+\left(\tfrac{17}{5}\right)^2+\left(\tfrac{21}{5}\right)^2+5^2\right)+4$$
$$= \frac{1}{25}\left(\frac{321}{5}\right)+4$$
$$= \frac{821}{125}$$

(d) Sketch the graph of a function $y = g(x)$, between $x = 1$ and $x = 5$, for which the area between the curve, the x-axis and the lines $x = 1$ and $x = 5$ is less than $\sum_{n=0}^{4}(g(1+nh))h$. [1]



Suggested Answers

3 **(a)** The function h is defined by $\mathrm{h} : x \mapsto \frac{1}{2}x^2 + 3$, for $x \in \mathbb{R}$.

The function g is defined by $\mathrm{g} : x \mapsto \dfrac{x+1}{5x-1}$, for $x \in \mathbb{R}$, $x \neq 0.2$.

(i) Find gh(2). [2]

$h(2) = 2+3 = 5$

$gh(2) = g(5) = \frac{6}{24} = \frac{1}{4}$

(ii) Find the value of x for which $\mathrm{g}(x) = 1.4$. [1]

$1.4 = \frac{x+1}{5x-1}$

$7x - 1.4 = x+1$

$6x = 2.4$

$x = 0.4$

(b) The function f is defined by $\mathrm{f} : x \mapsto \dfrac{x+a}{2x+b}$, for $x \in \mathbb{R}$, $x \neq k$.

(i) Give an expression for k and explain why this value of x has to be excluded from the domain of f. [2]

$2x+b = 0$

$x = -\frac{b}{2}$

$\therefore k = -\frac{b}{2}$

f is undefined at $x = -\frac{b}{2}$

The function f is such that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of f.

(ii) Determine the possible values of a and of b. [3]

$$2x+b \overset{\frac{1}{2}}{\overline{\big) x+a}}$$
$$\underline{x + b/2}$$
$$a - b/2$$

$y = \frac{1}{2}$ or $x = -\frac{b}{2}$

$\therefore R_f = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

$D_f = (-\infty, -\frac{b}{2}) \cup (-\frac{b}{2}, \infty)$

$b = -1$ $\quad a - \frac{b}{2} \neq 0$

$a \neq \frac{b}{2}$

$a \neq -\frac{1}{2}$

$\therefore$ possible values : $b = -1$ $\quad a = 1$

(iii) Find an expression for $\mathrm{f}^{-1}(-4)$. [1]

$f^{-1}(-4)$

$= f(-4)$

$= \dfrac{-4+a}{-8+b}$, $b \neq 8$

4 Mrs Wong is the president of a swimming club. She devises a training programme for members of the club. Members swim 10 lengths of a swimming pool; the time taken to swim the first length is 40 seconds and the time taken to swim the last length is 25 seconds. The times taken for each of the 10 lengths are in arithmetic progression.

(a) Find the total time taken to swim 10 lengths using Mrs Wong's programme. [2]

$$S_{10} = \frac{10}{2}(40 + 25)$$

$$= 325 \text{ s}$$

One of the members of the club, Alfie, devises a different training programme. In Alfie's programme the time taken to swim the first length is 25 seconds and the time taken to swim the last length is 40 seconds. The times taken for each of the 10 lengths are in geometric progression.

Suzie swims 30 lengths. She swims 10 lengths using Mrs Wong's programme, then she swims 10 lengths taking 25 seconds for each length, and then she swims 10 lengths using Alfie's programme. The length of the pool is 35 m.

(b) Find Suzie's average speed for her swim of 30 lengths. [5]

$$T_{10} = 25(r)^9 = 40$$

$$r^9 = \frac{8}{5} \quad r = 1.05361$$

$$S_{10} = \frac{25(r^{10} - 1)}{r - 1}$$

$$= 319.798$$

Total time: $319.798 + 325 + 250$

$$= 894.798$$

$$\approx 895 \text{ s}$$

Total dist: $30(35)$

$= 1050$

Avg speed $= \frac{1050}{894.798}$

$= 1.1734$

≈ 1.17 m/s

(c) Determine whether, exactly 8 minutes after she starts to swim, Suzie is swimming away from or towards her starting point. [2]

8 mins $= 480s > 325s$

$480 - 325 = 155s$

$155 \div 25 = 6.2$

$\therefore$ swimming away

Suggested Answers

5 **(a)** Find $\int \tan^2 5x \, dx$. [2]

$= \int \sec^2 5x - 1 \, dx$

$= \frac{\tan 5x}{5} - x + c$

(b) Find $\int_0^b \sin \overset{4x}{2x} \sin \overset{6x}{3x} \, dx$. [3]

$A - B = 4x$
$A + B = 6x$

$A = 5x$
$B = x$

$= \int_0^b -\frac{1}{2}(\cos 5x - \cos x) \, dx$

$= \int_0^b -\frac{1}{2}\cos 5x + \frac{1}{2}\cos x \, dx$

$= \left[-\frac{\sin 5x}{10} + \frac{\sin x}{2} \right]_0^b$

$= -\frac{\sin 5b}{10} + \frac{\sin b}{2}$

(c) Find $\displaystyle\int_a^b \frac{1}{x\ln x}\,dx$, where $1 < a < b$. [3]

$$= \int_a^b \frac{\frac{1}{x}}{\ln x}\, dx$$

$$= \Big[\ln\left|\ln x\right|\Big]_a^b$$

$$= \ln(\ln b) - \ln(\ln a)$$

$$= \ln\left(\frac{\ln b}{\ln a}\right) \quad \#$$

(d) Use the substitution $u = 1 + e^{2x}$ to find $\displaystyle\int \frac{e^{2x}}{(1+e^{2x})^3}\,dx$. [3]

$$\frac{du}{dx} = 2e^{2x} = 2(u-1)$$

$$\int \frac{e^{2x}}{(1+e^{2x})^3}\,dx$$

$$= \int \frac{u-1}{u^3}\left(\frac{1}{2(u-1)}\right) du$$

$$= \frac{1}{2}\int \frac{1}{u^3}\,du$$

$$= \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C$$

$$= -\frac{1}{4}\left(\frac{1}{u^2}\right) + c$$

$$= -\frac{1}{4}\,\frac{1}{(1+e^{2x})^2} + c \quad \#$$

Section B: Probability and Statistics [60 marks]

6 A biased 5-sided spinner gives the scores 1, 2, 3, 4 and 5 with the probabilities shown in the table, where p and q are constants.

Score	1	2	3	4	5
Probability	0.2	0.3	p	p	q

Given that the variance of the score is 1.61, calculate the mean score. [7]

$0.5 + 2p + q = 1$

$2p + q = 0.5 \Rightarrow q = 0.5 - 2p$

Let X be random variable denoting score on spinner

$E(X) = 0.2 + 0.6 + 3p + 4p + 5q$

$= 0.8 + 7p + 5q$

$= 3.3 - 3p$

$E(X^2) = 0.2 + 1.2 + 9p + 16p + 25q$

$= 1.4 + 25p + 25q$

$= 13.9 - 25p$

$Var(X) = 13.9 - 25p - (3.3 - 3p)^2$

$= 13.9 - 25p - (10.89 - 19.8p + 9p^2)$

$= -9p^2 - 5.2p + 3.01$

$\Rightarrow -9p^2 - 5.2p + 3.01 = 1.61$

$p = \frac{1}{5}$ or $-\frac{7}{9}$ rej $\because p > 0$

$\therefore E(X) = 3.3 - 3(\frac{1}{5})$

$= 2.7$ #

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Suggested Answers

7 When performing a trick a magician says the word ABRACADABRA. The 11 letters of this word are arranged in a row.

(a) Find the number of different arrangements that can be made. [2]

A B R C D
A B R
A
A
A

$$\frac{11!}{5!2!2!} = 83160$$ *

(b) Find the number of different arrangements in which the 2 B's are next to each other, the 2 R's are next to each other, exactly 4 of the A's are next to each other, and the C is next to the D. [3]

$\underline{AAAA}$ $\underline{BB}_\uparrow$ $\underline{RR}_\uparrow$ $\underline{CD}_\uparrow$

$4!\ {}^3C_1\ 2! = 144$

(c) Given that the 11 letters are arranged randomly, find the probability that all 5 A's are together. [3]

$\underline{AAAAA}$ _ _ _ _ _ _

$$\frac{7!}{2!2!} = 1260$$

$$\text{Prob} = \frac{1260}{83160} = \frac{1}{66}$$

8 A car manufacturer claims that the front tyres on a particular model of car have an average life span of 20 000 miles. Following comments from customers, the sales manager wishes to test if the life span of the tyres is greater than 20 000 miles.

(a) Explain why the sales manager should carry out a 1-tail test. State hypotheses for the test, defining any symbols you use. [3]

Because he want to test if the lifespan is greater than 20000

H_0 : null hypothesis

H_1 : Alternate hypothesis

μ : population mean life span of tyres

$H_0 : \mu = 20\,000$

$H_1 : \mu > 20\,000$

The sales manager contacts customers and gathers details about the life spans of a random sample of 50 of these tyres. The life spans, x thousand miles, are summarised below.

$$\Sigma(x-20) = 9.4 \qquad \Sigma(x-20)^2 = 38.76$$

(b) Calculate unbiased estimates of the population mean and variance of the life spans of the tyres. [2]

$\bar{x} = \frac{9.4}{50} + 20$

$= 20.188$

$s^2 = \frac{1}{49}\left(38.76 - \frac{9.4^2}{50}\right)$

$= 0.754955$

#1 Math Tuition

(c) Test, at the 5% level of significance, whether the mean life span of front tyres is more than 20 000 miles. [3]

$H_0 : \mu = 20000$

$H_1 : \mu > 20000$

Under H_0, by CLT $\bar{X} \sim N\left(20, \frac{0.7549}{50}\right)$ approx.

Test statistic : $Z = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim N(0,1)$

$= \frac{20.188 - 20}{\sqrt{0.7549}/\sqrt{50}}$

$= 1.53$

From G.C. p-value : $0.06300 > 0.05$

Do not rej H_0.

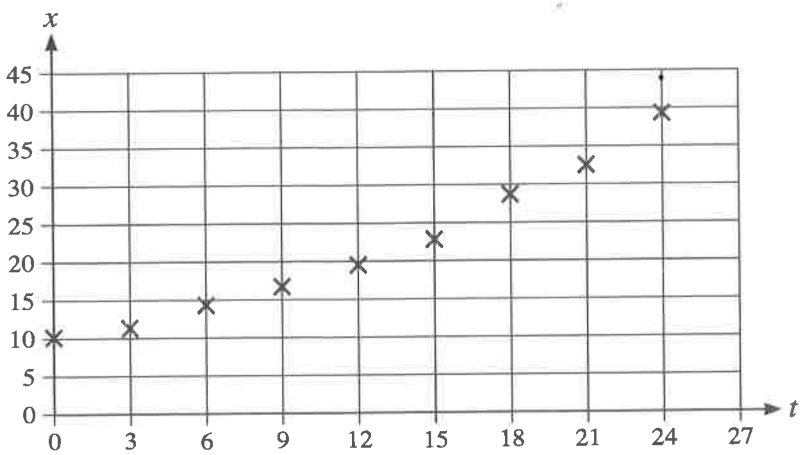
(d) Explain why this test would be inappropriate if the sales manager had taken a random sample of 15 tyres. [1]

15 is not large enough for the life span of tyres to be approximate to normal distribution using Central Limit Theorem

9 In a chemical reaction the mass, x grams, of a particular product at time t minutes is given in this table.

t	0	3	6	9	12	15	18	21	24
x	10.1	11.3	14.3	16.7	19.5	22.8	28.7	32.5	39.3

The value of the product moment correlation coefficient is 0.9803 correct to 4 significant figures. The scatter diagram for the data is shown below.



(a) Toby attempts to model the relationship between x and t with a straight line. Explain whether this is likely to provide a good model. [1]

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Toby now tries a model in which x has been transformed to $\ln x$.

(b) **(i)** Sketch a scatter diagram of $\ln x$ against t for the data given in the table. [1]

(ii) Toby models the data with the equation $\ln x = c + dt$. Find the values of the constants c and d and state the value of the product moment correlation coefficient for this model. [3]

(c) Comment on Toby's two models. [2]

10 In this question you should state the parameters of any normal distributions you use.

A company makes 3-legged wooden stools from 4 solid components – a seat in the form of a disc, and 3 legs each in the form of a long, thin cylinder. The seats and legs are bought in bulk from another company. Over a period of time it is found that the masses of the seats are normally distributed; 80% of the seats have mass less than 2.1 kg, and 15% of the seats have mass less than 1.95 kg.

(a) Find the mean mass of the seats and show that the standard deviation is 0.0799 kg, correct to 3 significant figures. [3]

Let X be random variable, denoting the mass of the seat in kg

Let mean mass of the seats be μ and standard deviation be σ

$X \sim N(\mu, \sigma)$

$P(X < 2.1) = 0.8$

$P(Z < \frac{2.1-\mu}{\sigma}) = 0.8$

$\frac{2.1-\mu}{\sigma} = 0.8416$

$2.1 = 0.8416\sigma + \mu$

$P(X < 1.95) = 0.15$

$P(Z < \frac{1.95-\mu}{\sigma}) = 0.15$

$\frac{1.95-\mu}{\sigma} = -1.0364$

$1.95 = -1.0364\sigma + \mu$

$\sigma = 0.07987$

≈ 0.0799

Shown

$\mu = 2.0327$

≈ 2.03 kg #

The masses of the legs, in kg, follow the distribution $N(1.2, 0.02^2)$.

(b) Find the expected number of legs with mass more than 1.21 kg in a randomly chosen batch of 500 legs. [2]

Let L be random variable denoting the mass of the legs in kg

$L \sim N(1.2, 0.02^2)$

$P(L > 1.21) = 0.30853$

Expected # of legs : 154.2

≈ 154 legs

(c) Find the probability that the total mass of a randomly chosen seat and 3 randomly chosen legs is between 5.6 kg and 5.7 kg. [3]

$X \sim N(2.0327,\ 0.0799^2)$

$L \sim N(1.2,\ 0.02^2)$

$X + L_1 + L_2 + L_3 \sim N(5.6327,\ 0.087086^2)$

$P(5.6 < X + L_1 + L_2 + L_3 < 5.7)$

$= 0.4265$

≈ 0.427 #

In order to make the stools, circular holes are drilled in the seats and the legs are fitted into them. In this process, the mass of seats is modelled as being reduced by 9% and the masses of the legs are unchanged.

(d) Find the probability that the total mass of a randomly chosen drilled seat and 3 randomly chosen legs is less than 5.6 kg. [3]

$0.91X \sim N(1.8497,\ 0.07271^2)$

$0.91X + L_1 + L_2 + L_3 \sim N(5.4497,\ 0.08054^2)$

$P(0.91X + L_1 + L_2 + L_3 < 5.6)$

$= 0.96899$

≈ 0.969 #

10 [Continued]

The holes made in the seats have diameters, in mm, that follow the distribution $N(31, 0.4^2)$ and the diameters of the legs, in mm, follow the distribution $N(30.7, 0.3^2)$. If the diameter of a leg is greater than the diameter of a hole, then the leg has to be sanded down to make it fit. If the diameter of a hole is more than 0.8 mm greater than the diameter of a leg, then padding has to be added when the leg is glued to the seat.

(e) A stool is made of a randomly chosen drilled seat and 3 randomly chosen legs. The legs are paired up with the holes at random. Find the probability that the 3 legs can be fitted without the need for any sanding or padding. [4]

Let H be random variable denoting the diameter of the holes in the seats

$H \sim N(31, 0.4^2)$

Let D be random variable denoting the diameter of the legs

$D \sim N(30.7, 0.3^2)$

$H - D \sim N(0.3, 0.5^2)$

$P(0 < H - D < 0.8)$

$= 0.567$

$\therefore$ Req prob $= 0.567^3$

$= 0.18237$

≈ 0.182 #

11 A company manufactures a wide variety of components for use in domestic appliances.

(a) The company introduces a new type of light fitting for refrigerators. Each day a supervisor selects a sample of 100 of the light fittings for testing.

(i) How should the light fittings be selected? Give a reason for this method of selection. [2]

He can select 1 light fitting in every 5 light fitting such that the prob of each light fitting is equal and constant throughout the selection process.

The supervisor records the number of faulty light fittings found on each of 150 working days. Her results are shown in the table.

L_1	Number faulty	0	1	2	3	4	5	6	7	8 or more
L_2	Number of days	4	19	38	41	22	16	6	4	0

(ii) Use the information in the table to estimate p, the probability that a light fitting is faulty. [1]

use gc 1-var test

$\bar{x} = 3$

$\therefore p = 3/100$

$= 0.03$

(iii) Assuming that the number of faulty fittings found each day follows the binomial distribution $B(100, p)$, find the expected number of days on which 3 faulty fittings are found in a period of 150 working days. [2]

Let F be r.v denoting # of faulty fittings

$F \sim B(100, 0.03)$

$P(F=3) = 0.22747$

Expected # of days $= 34.1$

≈ 34 days

11 [Continued]

(b) The company also makes heating elements for electric ovens. A fixed number of randomly chosen heating elements are tested each day and the number found to be faulty is denoted by X.

(i) State, in context, two assumptions needed for X to be well modelled by a binomial distribution. [2]

Event of each heating element found faulty is independent of another heating element.

Prob of a heating element found faulty is constant throughout the sample

Assume now that X has the distribution $B(80, 0.02)$.

(ii) Find the probability that, on a randomly chosen day, the number of elements found to be faulty is between 1 and 4 inclusive. [2]

$X \sim B(80, 0.02)$

$P(1 \leqslant X \leqslant 4) = 0.7789$

≈ 0.779 #

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(iii) Find the probability that, in a randomly chosen 5-day working week, more than 3 elements are found to be faulty on at least 2 days. [3]

$X \sim B(80, 0.02)$

$P(X > 3) = 1 - P(X \leq 3)$

$= 0.076854$

Let Y be random variable denotin more than 3 faulty elements out of 5 days

$Y \sim B(5, 0.076854)$

$P(Y \geq 2) = 1 - P(Y \leq 1)$

$= 0.050499$

≈ 0.0505 #

(iv) Find the probability that, in a randomly chosen 5-day working week, no more than 8 faulty elements are found in total. [2]

Let W be r.v, denoting # of faulty elements in 5days

$W \sim B(400, 0.02)$

$P(W \leq 8) = 0.5925$

≈ 0.593 #